

# Non-exponential relaxations in disordered conductors

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We show that, in low dimensional conductors, the quasiparticle decay and the relaxation of the phase are not exponential processes. In quasi-one dimension, they scale as  $e^{-(t/\tau_N)^{3/2}}$  where the characteristic time  $\tau_{in}$ , identical for both processes, is a power  $T^{2/3}$  of the temperature. This result implies a distribution of relaxation times.

The problem of dephasing in the presence of electron-electron interaction in disordered conductors was first addressed in the pioneering work of Altshuler and Aronov<sup>1</sup>. It has been revived by a series of recent experiments<sup>2,3,4</sup>. Another very related question is the understanding of the time evolution of a quasiparticle state which governs the relaxation towards equilibrium. This quasiparticle decay is usually described by a characteristic time, implying implicitly an exponential process. Here we show that in low dimension, and in particular in quasi-one-dimensional disordered wires, this decay is faster than exponential. We focus on two aspects, the decay of a quasiparticle state and the loss of phase coherence. It has been shown that both processes are related<sup>5</sup>. They are described by the same characteristic time scale. Here we show that *both* processes are non exponential<sup>6,7</sup>.

We start by considering the lifetime of a quasiparticle state, initially in a state of energy  $\epsilon$ . Its interaction with another quasiparticle of energy  $\epsilon'$  leads to final states at energies  $\epsilon - \omega$  and  $\epsilon' + \omega$ . At zero temperature, the lifetime is given by

$$\frac{1}{\tau_{ee}(\epsilon)} = 4\pi\nu_0^3 \int_0^\epsilon \omega K(\omega) d\omega \quad (1)$$

where  $\nu_0$  is the density of states per spin direction. The kernel  $K(\omega)$  is the disordered averaged squared matrix element of the screened Coulomb interaction. It depends only on the energy transfer  $\omega$  and it can be written as<sup>1</sup>

$$K(\omega) = \frac{1}{\pi^2\nu_0^2\Omega^2} \sum_{\mathbf{q} \neq 0} |U(\mathbf{q}, \omega)|^2 [\text{Re}P_d(\mathbf{q}, \omega)]^2 \quad (2)$$

In this expression  $P(\mathbf{q}, \omega) = 1/(-i\omega + Dq^2)$  is the classical diffusion probability,  $U(\mathbf{q}, \omega)$  is the dynamically screened coulomb potential  $U(\mathbf{q}, \omega) = (1/2\rho_0)Dq^2/(-i\omega + Dq^2)$  and  $\rho_0$  is the density of states per spin direction and per volume unit. The kernel can also be written under the form

$$K(\omega) = \frac{1}{4\pi^2\nu_0^4} \sum_{\mathbf{q}} \frac{1}{\omega} \text{Im}P_d(\mathbf{q}, \omega) = \frac{1}{4\pi^2\nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{1}{\omega^2 + D^2q^4} \quad (3)$$

so that the lifetime reduces to

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{1}{\pi\nu_0} \int_0^\epsilon d\omega \sum_{\mathbf{q}} \text{Im}P_d(\mathbf{q}, \omega) \quad (4)$$

Replacing the discrete sum over  $\mathbf{q}$  by an integral leads to

$$K(\omega) = \frac{\alpha_d}{16\nu_0^4\omega^2} \left( \frac{\omega}{E_c} \right)^{d/2} \quad (5)$$

where  $\alpha_1 = \sqrt{2}/\pi^2$ ,  $\alpha_2 = 1/2\pi^2$ ,  $\alpha_3 = \sqrt{2}/2\pi^3$  and  $E_c = D/L^2$  is the Thouless energy. From Eq. (1), one obtains the well-known<sup>1</sup> power-law dependence of the lifetime  $1/\tau_{ee}(\epsilon) \propto \epsilon^{d/2}$ .

At first sight, one could consider that, at finite temperature, the quasiparticle relaxation rate is given by  $1/\tau_{ee}(\epsilon, T) \propto \max(\epsilon^{d/2}, T^{d/2})$ . This is not correct for  $d \leq 2$  because of the importance of the processes with low energy transfer<sup>1</sup>. However, it has been found that the quasiparticle relaxation is still exponential with a characteristic time which varies like  $T^{2/3}$  in one dimension<sup>1</sup>. Here we show that in low dimensions  $d \leq 2$ , *the relaxation is not exponential*. To that purpose, we start from the expression

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\epsilon' F(\epsilon, \epsilon', \omega) K(\omega) \quad (6)$$

that generalizes Eq. (1) to finite temperatures. The thermal function  $F(\epsilon, \epsilon', \omega)$  depends on the Fermi factors  $f_\epsilon = 1/(e^{\beta\epsilon} + 1)$  and has the form

$$F(\epsilon, \epsilon', \omega) = f_{\epsilon'}(1 - f_{\epsilon-\omega})(1 - f_{\epsilon'+\omega}) + (1 - f_{\epsilon'})f_{\epsilon-\omega}f_{\epsilon'+\omega} \quad (7)$$

The first term is larger for  $\epsilon > 0$ , it describes the relaxation of an electronic state above the Fermi level. The second term, larger for  $\epsilon < 0$ , accounts for the relaxation of a hole state within the Fermi sea. The two terms are equal for  $\epsilon = 0$ . Integrating over  $\epsilon'$ , and after some straightforward manipulations, it rewrites

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 2\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega K(\omega) \omega \left( \coth \frac{\beta\omega}{2} + \tanh \frac{\beta}{2}(\epsilon - \omega) \right) \quad (8)$$

This expression can also be obtained from the imaginary part of the self-energy, a calculation that incorporates implicitly both processes involved in Eq. (7)<sup>8</sup>.  $\tau_{ee}(\epsilon, T)$  is defined as the lifetime of a quasiparticle. It can also be interpreted as being the characteristic time that describes the relaxation towards equilibrium. Indeed, it can be defined from a Boltzmann equation for the energy distribution  $n_\epsilon$ <sup>1,9</sup>

$$\frac{\partial n_\epsilon}{\partial t} = -4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega K(\omega) \int_{-\infty}^{\infty} d\epsilon' [n_\epsilon n_{\epsilon'}(1 - n_{\epsilon-\omega})(1 - n_{\epsilon'+\omega}) - n_{\epsilon-\omega} n_{\epsilon'+\omega}(1 - n_\epsilon)(1 - n_{\epsilon'})] \quad (9)$$

The relaxation term contains two contributions that respectively describe the quasiparticles leaving a given quantum state ("out" term) and incoming in this state ("in"). At equilibrium  $n_\epsilon$  equals the Fermi factor  $f_\epsilon = 1/(e^{\beta\epsilon} + 1)$  and the relaxation term is zero. Linearising around the equilibrium distribution  $n_\epsilon = f_\epsilon + \delta n_\epsilon$ , one obtains a relaxation of the form  $\partial \delta n_\epsilon / \partial t = -\delta n_\epsilon / \tau_{ee}(\epsilon, T)$  with the same characteristic time given by (6). The lifetime of a quasiparticle state can also be interpreted as the relaxation time of the energy distribution.

We now consider the lifetime of a quasiparticle *at the Fermi level* and we denote it by  $\tau_{in}(T)$ . From Eq. (8), it is given by<sup>10</sup>

$$\frac{1}{\tau_{in}(T)} = \frac{1}{\tau_{ee}(0, T)} = 8\pi\nu_0^3 \int_0^{\infty} d\omega K(\omega) \frac{\omega}{\sinh \beta\omega} \quad (10)$$

In  $d = 3$ , the integral converges, leading to the well-known  $T^{3/2}$  behavior<sup>1</sup>. For  $d \leq 2$ , the integral diverges at low energy transfer. It has been argued that the integral has to be cut-off at low energy: since the lifetime of a quasiparticle is finite, no energy transfer can be smaller than this inverse lifetime<sup>1</sup>. Consequently, the lifetime is solution of the self-consistent equation (for simplicity the thermal factor in Eq. (10) has been replaced by an upper cut-off at  $\omega \sim T$ ) :

$$\frac{1}{\tau_{in}(T)} \simeq \frac{T}{\nu_0} \int_{1/\tau_{in}(T)}^T \frac{d\omega}{\omega^2} \left( \frac{\omega}{E_c} \right)^{d/2} \quad (11)$$

This argument gives a characteristic time  $\tau_{in}(T)$  which scales as  $T^{2/3}$  in one dimension :

$$\frac{1}{\tau_{in}(T)} \sim \left( \frac{\Delta T}{E_c^{1/2}} \right)^{2/3}. \quad (12)$$

$\Delta = 1/\nu_0$  is the interlevel energy spacing.

We argue that the divergence at low energy transfer in (11) has a deeper significance<sup>6</sup>. It is the signature of a *non-exponential relaxation*. The reason goes as follows: the quasiparticle state decay cannot be exponential since, after a time  $t$ , the energy transfer cannot be defined with a precision better than  $1/t$  (Heisenberg inequality). Thus *the decay rate cannot be constant in time*<sup>6</sup>. Instead of assuming a relaxation of the initial state of the form  $\mathcal{P}(t) = e^{-t/\tau_{in}}$  with eq. (11), we find that the probability  $\mathcal{P}(t)$  for the quasiparticle to stay in its initial state is given by

$$-\ln \mathcal{P} \simeq \frac{Tt}{\pi\nu_0} \int_{1/t}^T \frac{d\omega}{\omega^2} \left( \frac{\omega}{E_c} \right)^{d/2}, \quad (13)$$

an expression valid for times  $t \gg 1/T$ . In one dimension, one has<sup>7</sup>:

$$-\ln \mathcal{P} = \frac{Tt}{\sqrt{2}\pi\nu_0} \int_{1/t}^T \frac{d\omega}{\omega^2} \left( \frac{\omega}{E_c} \right)^{1/2} = \frac{\sqrt{2}T}{\pi\nu_0\sqrt{E_c}} t^{3/2} \quad (14)$$

which leads to a non-exponential behavior for the relaxation of a quasiparticle state.

$$\mathcal{P}(t, T) \sim e^{-[t/\tau_{in}(T)]^{3/2}} \quad d = 1 \quad (15)$$

with the same characteristic time as in Eq. (12). Similarly, in two dimensions, starting from (13), one finds a logarithmic correction to the exponential decrease

$$\mathcal{P}(t, T) \sim e^{-\frac{t}{\tau_{in}} \frac{1}{\ln Tt}} \quad d = 2 \quad (16)$$

where  $\tau_{in}(T) \propto \Delta T/E_c$ .

Now, one can wonder whether this peculiar behavior of the relaxation of a quasiparticle state appears in the time dependence of other quantities. To that purpose, we consider the phase coherence time. It is defined as the lifetime of the Cooperon in the presence of other electrons. The Cooperon is the quantum correction to the return probability that consists in pairs of time reversed trajectories. Altshuler, Aronov and Khmelnitskii<sup>12</sup> have shown that the effect of other electrons can be accounted for by a fluctuating electric potential  $V(\mathbf{r}, \tau)$  whose characteristics are given by the fluctuation-dissipation theorem:

$$\langle V(\mathbf{r}, \tau) V(\mathbf{r}', \tau') \rangle_T = \frac{\delta(\tau - \tau')}{(2\pi)^d} \frac{2T}{\sigma_0} \int \frac{d\mathbf{q}}{q^2} e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \quad (17)$$

The Cooperon contribution to the return probability takes the form

$$P_c(\mathbf{r}, \mathbf{r}, t) = P_c^{(0)}(\mathbf{r}, \mathbf{r}, t) \left\langle e^{i\Phi(\mathbf{r}, t)} \right\rangle_{T, \mathcal{C}} \quad (18)$$

where  $P_c^{(0)}$  is the cooperon in the absence of the fluctuating potential.  $\Phi = \Phi(\mathbf{r}, t)$  is the relative phase for a pair of time reversed trajectories at time  $t$  :

$$\Phi = \frac{e}{\hbar} \int_0^t [V(\mathbf{r}(\tau), \tau) - V(\mathbf{r}(\tau), \bar{\tau})] d\tau \quad (19)$$

One of the trajectories propagates from time  $\tau = 0$  to  $\tau = t$ , while the time reversed trajectory propagates from  $\tau = t$  to  $\tau = 0$ . We define  $\bar{\tau} = t - \tau$  and  $\langle \cdots \rangle_{T,C}$  is the average taken both on the distribution of diffusion trajectories ( $\langle \cdots \rangle_C$ ) and on the thermal fluctuations ( $\langle \cdots \rangle_T$ ) of the electric potential. The thermal fluctuations are gaussian so that the thermal average  $\langle e^{i\Phi} \rangle_T$  satisfies :

$$\langle e^{i\Phi} \rangle_T = e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} . \quad (20)$$

Let us start with the calculation of  $\langle \Phi^2 \rangle_T$ . Inserting (17) into (19), one finds :

$$\langle \Phi^2 \rangle_T = \frac{4e^2T}{\sigma_0\hbar^2} \int_0^t d\tau \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} [1 - \cos \mathbf{q} \cdot (\mathbf{r}(\tau) - \mathbf{r}(\bar{\tau}))] . \quad (21)$$

In one dimension, the phase fluctuation becomes

$$\langle \Phi^2 \rangle_T = \frac{2e^2T}{\hbar^2\sigma_0S} \int_0^t |r(\tau) - r(\bar{\tau})| d\tau . \quad (22)$$

It depends on the trajectory  $r(\tau)$  and we still need to calculate the average

$$\left\langle e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} \right\rangle_C \quad (23)$$

over the distribution of diffusive trajectories. A first approximation consists in assuming that  $\left\langle e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} \right\rangle_C = e^{-\frac{1}{2}\langle \Phi^2 \rangle_{T,C}}$ . Averaging over the diffusive trajectories, we obtain that  $|r(\tau) - r(\bar{\tau})|$  scales as  $\sqrt{t}$  so that, from (22), one has

$$\langle \Phi^2 \rangle_{T,C} = \frac{\sqrt{\pi}e^2T}{2\hbar^2\sigma_0S} \sqrt{D} t^{3/2} = \frac{\sqrt{\pi}}{2} \left( \frac{t}{\tau_N} \right)^{3/2} \quad (24)$$

where we have defined the characteristic time

$$\tau_N = \left( \frac{\hbar^2\sigma_0S}{e^2T\sqrt{D}} \right)^{2/3} \quad (25)$$

so that we find, within this approximation :

$$\langle e^{i\Phi} \rangle_{T,C} \simeq e^{-\sqrt{\pi}/4(t/\tau_N)^{3/2}} . \quad (26)$$

A similar behavior has been found in<sup>11</sup>. Apart from a numerical factor, the characteristic time (25) is the same as the inelastic time (12) for the decay of a quasiparticle state. Moreover Eq. (24) shows that the phase relaxation has the same temporal dependence as the quasiparticle relaxation (15).

However, the result (26) is not yet fully correct since we have made an approximation in replacing the average  $\langle \cdots \rangle$  of the exponential by the exponential of the average. The exact behavior of the phase relaxation (23) can be obtained from a functional integral approach<sup>12</sup> that shows that the Laplace transform

$$P_\gamma(r, r) = \int dt P_c(r, r, t) e^{-\gamma t}$$

is given by :

$$P_\gamma(r, r) = -\frac{1}{2} \sqrt{\frac{\tau_N}{D}} \frac{\text{Ai}(\tau_N/\tau_\gamma)}{\text{Ai}'(\tau_N/\tau_\gamma)} \quad (27)$$

with Ai et Ai' being respectively the Airy function and its derivative<sup>13</sup> and  $\tau_\gamma = 1/\gamma$ . The probability  $P_c(r, r, t)$  could be obtained from inverse Laplace transform of Eq. (27). Its time dependence is clearly non exponential. Here we obtain the time dependence of the phase relaxation.

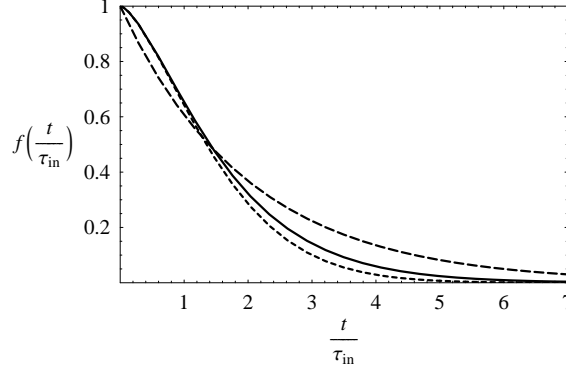


Figure 1: Behaviour of  $\langle e^{i\Phi(t)} \rangle_{T,C}$ . The continuous line is the exact result (29). The dotted line is obtained from the small time expansion (26). The dashed line shows the exponential fit  $e^{-t/2\tau_{in}}$ .

It is of the form  $\langle e^{i\Phi} \rangle_{T,C} = f(t/\tau_N)$ . Since in one dimension, one has  $P_c^{(0)}(\mathbf{r}, \mathbf{r}, t) = 1/\sqrt{4\pi Dt}$ , the unknown function  $f(t/\tau_N)$  is solution of

$$\int_0^\infty \frac{1}{\sqrt{4\pi Dt}} f\left(\frac{t}{\tau_N}\right) e^{-t/\tau_\gamma} dt = -\frac{1}{2} \sqrt{\frac{\tau_N}{D}} \frac{\text{Ai}(\tau_N/\tau_\gamma)}{\text{Ai}'(\tau_N/\tau_\gamma)} \quad (28)$$

The inverse Laplace transform is obtained by noticing that both the Airy function and its derivative are analytic and non meromorphic functions whose zeroes lie on the negative real axis. Then, by performing the integral in the complex plane with the residues  $\text{Res}(e^{st} \text{Ai}(s)/\text{Ai}'(s)) = e^{-|u_n|t}/|u_n|$  where the  $u_n$  are the zeros of  $\text{Ai}'(s)$  given at a very good approximation by  $|u_n| = (\frac{3\pi}{2}(n - \frac{3}{4}))^{2/3}$  <sup>13</sup>, we obtain the analytic function

$$\langle e^{i\Phi} \rangle_{T,C} = \sqrt{\frac{\pi t}{\tau_{in}}} \sum_{n=1}^{\infty} \frac{e^{-|u_n|t/\tau_{in}}}{|u_n|} \quad (29)$$

At small times  $t < \tau_{in}$ , it behaves like eq. (26). At large time, the relaxation is driven by the first zero of the  $\text{Ai}'$  function, namely  $\langle e^{i\Phi} \rangle_{T,C} \simeq \sqrt{\pi t/\tau_{in}} e^{-|u_1|t/\tau_{in}}/|u_1|$  with  $|u_1| \simeq 1.019$ . Clearly, the relaxation (29) is never exponential. It appears as a distribution of relaxation times  $\tau_{in}/|u_n|$  which is at the origin of the rather unexpected compressed exponential behavior of the quasiparticle decay and of the Cooperon phase relaxation.

In conclusion, we have shown that the decay of a quasiparticle state in a low dimensional disordered conductor is not exponential. In quasi-1d, it is of the form  $e^{-(t/\tau_{in})^{3/2}}$ . We have also calculated the relaxation of the phase in a quasi-1d conductor and we have shown that it is also described by the same time dependence as for the relaxation of the energy, with the same characteristic time.

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